Rezolvare Bacalaureat 2025 – Matematică M1

Subiectul I

1. $z_1 = 1-i$, $z_2 = 2+i$

$$2z_1 + iz_2 = 2(1-i) + i(2+i) = 2-2i+2i+i^2 = 2-1 = 1.$$

2. $f(x) = x+3$ | $f(a) = a+3$

$$(f\circ f)(a) = f(f(a)) = f(a+3) = a+6.$$

$$a+6 = 9 \iff a = 3$$

3. $\sqrt{2x^2-3x+2} = x$

$$\text{C.E.}: \begin{cases} 2x^2-3x+2 \geq 0 \\ x \geq 0 \end{cases}$$

$$\left.\begin{array}{l} 2x^2-3x+2 = 0 \\ \Delta = 9-16 < 0 \end{array}\right\} \Rightarrow 2x^2-3x+2 \geq 0 \quad \forall x \in \mathbb{R}$$

$\Rightarrow D: [0, +\infty)$

$$2x^2-3x+2 = x^2 \iff x^2-3x+2 = 0$$

$$(x-1)(x-2) = 0$$

$\Rightarrow x = 1$ sau $x = 2$

$\Rightarrow x \in \{1, 2\}$

4. Cazuri posibile : $99 - 10 + 1 = 90$

Cazuri favorabile : $D_{64} \cap \mathbb{N} = \{1, 2, 4, 8, 16, 32, 64\}$

$\Rightarrow$ Avem 3 cazuri fav.

$$P = \frac{\text{cazuri fav.}}{\text{cazuri pos.}} = \frac{3}{90} = \frac{1}{30}.$$

5. $A(0,1)$, $B(5,0)$, $C(6,3)$, $D(a,a)$

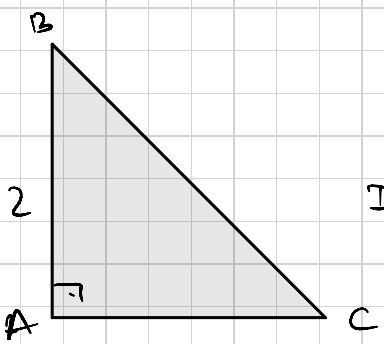
Fie M – mijl. lui AC

$$\Rightarrow \begin{cases} x_M = \dfrac{x_A + x_C}{2} \\ y_M = \dfrac{y_A + y_C}{2} \end{cases} \Leftrightarrow \begin{cases} x_M = 3 \\ y_M = 2 \end{cases}$$

Dar, M – mijl. lui BD $\Rightarrow \begin{cases} x_M = \dfrac{x_B + x_D}{2} \\ y_M = \dfrac{y_B + y_D}{2} \end{cases}$

$$\Leftrightarrow \begin{cases} 3 = \dfrac{5 + x_D}{2} \\ 2 = \dfrac{0 + y_D}{2} \end{cases} \Leftrightarrow \begin{cases} x_D = 1 \\ y_D = 4 \end{cases} \Rightarrow \underline{D(1,4)}$$

6.



$tg\, B = 3 \Rightarrow \dfrac{AC}{AB} = 3$

$\Rightarrow AC = 6$

Din T. Pitagora $\Rightarrow AB^2 + AC^2 = BC^2$

$4 + 36 = BC^2$

$\Rightarrow BC^2 = 40 \Rightarrow \underline{BC = 2\sqrt{10}}$

Subiectul II

1. $A(x) = \begin{pmatrix} 2-3x & 0 & x \\ 0 & 2 & 0 \\ -9x & 0 & 3x+2 \end{pmatrix}$

a) $A(1) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ -9 & 0 & 5 \end{pmatrix}$

$\Rightarrow \det(A(1)) = \begin{vmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ -9 & 0 & 5 \end{vmatrix} = -10 + 0 + 0 + 18 - 0 - 0 = 8.$

b) $A(x) \cdot A(y) =$

$= \begin{pmatrix} 2-3x & 0 & x \\ 0 & 2 & 0 \\ -9x & 0 & 3x+2 \end{pmatrix} \cdot \begin{pmatrix} 2-3y & 0 & y \\ 0 & 2 & 0 \\ -9y & 0 & 3y+2 \end{pmatrix}$

$= \begin{pmatrix} (2-3x)(2-3y) - 9xy & 0 & y(2-3x) + x(3y+2) \\ 0 & 4 & 0 \\ -9x(2-3y) - 9y(3x+2) & 0 & -9xy + (3x+2)(3y+2) \end{pmatrix}$

$$= \begin{pmatrix} 4-6y-6x & 0 & 2y - \cancel{3xy} + \cancel{3xy} + 2x \\ 0 & 4 & 0 \\ -18x-18y & 0 & 6x+6y+4 \end{pmatrix} \quad (1)$$

$$2\, A(x+y) = 2 \cdot \begin{pmatrix} 2-3x-3y & 0 & x+y \\ 0 & 2 & 0 \\ -9x-9y & 0 & 2+3x+3y \end{pmatrix}$$

$$= \begin{pmatrix} 4-6x-6y & 0 & 2x+2y \\ 0 & 4 & 0 \\ -18x-18y & 0 & 6x+6y+4 \end{pmatrix} \quad (2)$$

$$(1) = (2) \Longrightarrow \quad A(x) \cdot A(y) = 2A(x+y) \quad \forall\, x,y \in \mathbb{R}$$

(qed)

c) $\left(A(x) + A(3x)\right) \cdot A(2x) = 4A(x^2)$

$$A(x) + A(3x) = \begin{pmatrix} 2-3x & 0 & x \\ 0 & 2 & 0 \\ -9x & 0 & 2+3x \end{pmatrix} + \begin{pmatrix} 2-9x & 0 & 3x \\ 0 & 2 & 0 \\ -27x & 0 & 2+9x \end{pmatrix}$$

$$= \begin{pmatrix} 4-12x & 0 & 4x \\ 0 & 4 & 0 \\ -36x & 0 & 4+12x \end{pmatrix} = 2 \begin{pmatrix} 2-6x & 0 & 2x \\ 0 & 2 & 0 \\ -18x & 0 & 2+6x \end{pmatrix}$$

$= 2A(2x)$

Relația este $\quad 2A(2x) \cdot A(2x) = 4A(x^2)$

Dim e) $\Rightarrow \quad A(2x) \cdot A(2x) = 2A(4x) \quad | \cdot 2$

$\Rightarrow \quad 2A(2x) \cdot A(2x) = 4A(4x) = 4A(x^2)$

$4x = x^2 \iff x^2 - 4x = 0$

$x(x-4) = 0 \Rightarrow x = 0$ sau $x = 4$

$\Rightarrow x \in \{0, 4\}$

2. $f = ax^3 + 3x^2 - ax - 6 \quad \forall a \in \mathbb{R}^*$

a) $f(1) = a + 3 - a - 6 = 3 - 6 = -3$

b) $a = 1 \Rightarrow f = x^3 + 3x^2 - x - 6$

$$\begin{array}{l|l} x^3 + 3x^2 - x - 6 & x^2 + 3x - 1 \\ \hline -x^3 - 3x^2 + x & x \\ \hline / \quad / \quad / \quad -6 & \end{array}$$

$\Rightarrow \quad x^3+3x^2-x-6 = x(x^2+3x-1)-6$

Câtul este x, restul este -6

c) $(1+x_1)(1+x_2)(1+x_3) =$

$= (1+x_1+x_2+x_1x_2)(1+x_3)$

$= 1+x_1+x_2+x_1x_2+x_3+x_1x_3+x_2x_3+x_1x_2x_3$

Viete: $\begin{cases} S_1 = x_1+x_2+x_3 = -\frac{3}{a} \\ S_2 = x_1x_2+x_1x_3+x_2x_3 = -\frac{a}{a} = -1. \\ S_3 = x_1x_2x_3 = \frac{6}{a} \end{cases}$

$\Rightarrow \quad 1+S_1+S_2+S_3 = 1$

$\Leftrightarrow \quad -\frac{3}{a}+\frac{6}{a}-1 = 0 \quad \Leftrightarrow \quad \frac{3}{a} = 1 \Rightarrow \underline{a=3}.$

Subiectul III

1. $f:(0,+\infty) \to \mathbb{R}, \quad f(x) = 2x+\ln\frac{x}{x+2}$

$= 2x+\ln x-\ln(x+2)$

a) $f'(x) = \left(2x+\ln x-\ln(x+2)\right)'$

$= 2+\frac{1}{x}-\frac{1}{x+2} = \frac{2x(x+2)+x+2-x}{x(x+2)}$

$$= \frac{2x^2+4x+2}{x(x+2)} = \frac{2(x+1)^2}{x(x+2)} \quad \forall\, x\in(0,+\infty)$$

b) $y = mx + n \quad , \quad m, n \in \mathbb{R}$

$$m = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{2x + \ln\frac{x}{x+2}}{x}$$

$$= \lim_{x\to\infty}\left(\frac{2\cancel{x}}{\cancel{x}} + \frac{\ln\left(\frac{x}{x+2}\right)}{x}\right)$$

$$= \lim_{x\to\infty}\left(2 + \frac{\ln\left(\frac{x}{x+2}\right)}{x}\right)$$

(*) obs: $\lim\limits_{x\to\infty} \frac{x}{x+2} = 1 \Rightarrow \lim\limits_{x\to\infty} \ln\left(\frac{x}{x+2}\right) = 0.$

$\Rightarrow m = 2.$

$$n = \lim_{x\to\infty} [f(x) - m\cdot x] = \lim_{x\to\infty}\left(\cancel{2x} + \ln\frac{x}{x+2} - \cancel{2x}\right)$$

$$= \lim_{x\to\infty} \ln\frac{x}{x+2} \overset{*}{=} 0.$$

$\Rightarrow y = 2x$ asimp. oblică spre $+\infty$ la G_f.

c) $f'(x) = \frac{2(x+1)^2}{x(x+2)} = 0 \Leftrightarrow 2(x+1)^2 = 0$

$\Leftrightarrow x = -1 \notin (0,+\infty)$

$f'(1) = \frac{2\cdot 4}{1\cdot 3} > 0$ și cum $f'(x) \neq 0 \;\; \forall\, x\in(0,\infty)$

$\Rightarrow \quad f'(x) > 0 \quad \forall\, x \in (0,+\infty) \quad \Rightarrow \quad f$ strict cresc.

$\Rightarrow \quad f$ monotonă $\quad \Rightarrow \quad f$ injectivă $\quad (1)$

$$\lim_{\substack{x\to 0\\ x>0}} f(x) = \lim_{\substack{x\to 0\\ x>0}} \left(\underbrace{2x}_{0} + \ln \frac{x}{x+2}\right)$$

$$= \lim_{\substack{x\to 0\\ x>0}} \ln \frac{x}{x+2} = \lim_{\substack{x\to 0\\ x>0}} \ln x = -\infty. \quad (2)$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(2x + \underbrace{\ln \frac{x}{x+2}}_{(*)}\right) = 2\cdot\infty = \infty. \quad (3)$$

Din $(1), (2), (3)$ şi f - continuă $\forall\, x \in (0,\infty)$

$\Rightarrow \quad \operatorname{Im} f = \mathbb{R} = $ codomeniu $\Rightarrow f$ surjectivă (4)

$\overset{(1),(4)}{\Longrightarrow} \quad f$ bijectivă.

2. $f : (-1,+\infty) \to \mathbb{R}, \quad f(x) = \dfrac{x^2}{(x+1)^3}$

a) $$\int_0^3 f(x)\cdot (x+1)^3\, dx = \int_0^3 x^2\, dx = \frac{x^3}{3}\Big|_0^3$$

$$= 3^2 - 0 = 9.$$

b) $$\int_0^1 \sqrt{f(x)(x+1)}\,dx = \int_0^1 \sqrt{\frac{x^2}{(x+1)^3}\cdot(x+1)}\,dx$$

$$= \int_0^1 \sqrt{\frac{x^2}{(x+1)^2}}\,dx = \int_0^1 \frac{x}{x+1}\,dx$$

$$= \int_0^1 \frac{x+1-1}{x+1}\,dx = \int_0^1 dx - \int_0^1 \frac{1}{x+1}\,dx$$

$$= \left(x - \ln(x+1)\right)\Big|_0^1 = 1 - \ln 2.$$

c) $$g : \mathbb{R} \to \mathbb{R},\quad g(x) = \frac{f(e^x)}{e^x} = \frac{e^{2x}}{(e^x+1)^3}\cdot\frac{1}{e^x}$$

$$= \frac{e^x}{(e^x+1)^3}$$

$$A = \int_{-1}^{1} |g(x)|\,dx = \int_{-1}^{1} \Bigg|\underbrace{\frac{e^x}{(e^x+1)^3}}_{>0}\Bigg|\,dx$$

$$\Rightarrow \quad A = \int_{-1}^{1} \frac{e^x}{(e^x+1)^3}\,dx$$

$$t = e^x + 1 \quad \begin{cases} x = -1 \Rightarrow t = e^{-1}+1 \\ x = 1 \Rightarrow t = e+1 \end{cases}$$

$$dt = e^x\,dx$$

$$\Rightarrow \quad A = \int_{e^{-1}+1}^{e+1} \frac{dt}{t^3} = \int_{e^{-1}+1}^{e+1} t^{-3}\, dt = \left.\frac{t^{-2}}{-2}\right|_{e^{-1}+1}^{e+1}$$

$$= -\left.\frac{1}{2t^2}\right|_{e^{-1}+1}^{e+1} = -\frac{1}{2(e+1)^2} + \frac{1}{2(e^{-1}+1)^2}$$

$$= \frac{1}{2\cdot\left(\frac{e+1}{e}\right)^2} - \frac{1}{2(e+1)^2} = \frac{e^2}{2(e+1)^2} - \frac{1}{2(e+1)^2}$$

$$= \frac{(e-1)(e+1)}{2(e+1)^2} = \frac{e-1}{2(e+1)} \quad (\text{qed.})$$